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Equivalence theorem, consistency and axiomatizations of a multi-choice value

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Abstract This paper is devoted to the study of solutions for multi-choice games which admit a potential, such as the potential associated with the extended Shapley value proposed by Hsiao and Raghavan (Int J Game Theory 21:301–302, 1992; Games Econ Behav 5:240–256, 1993). Several axiomatizations of the family of all solutions that admit a potential are offered and, as a main result, it is shown that each of these solutions can be obtained by applying the extended Shapley value to an appropriately modified game. In the framework of multi-choice games, we also provide an extension of the reduced game introduced by Hart and Mas-Colell (Econometrica 57:589–614, 1989). Different from the works of Hsiao and Raghavan (1992, 1993), we provide two types of axiomatizations, one is the analogue of Myerson's (Int J Game Theory 9:169–182, 1980) axiomatization of the Shapley value based on the property of balanced contributions. The other axiomatization is obtained by means of the property of consistency.

Keywords Multi-choice games · Shapley value · Potential · Balanced contributions · Consistency

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1 Introduction

A multi-choice TU game, introduced by Hsiao and Raghavan [5,6], is a generalization of a traditional TU game. In a traditional TU game, each player is either fully involved or not

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involved at all in participation with some other agents, while in a multi-choice game, each player is allowed to participate with finite many different activity levels. As we knew, solutions on multi-choice games could be applied in many fields such as economics, political sciences, accounting, and even military sciences.

The Shapley value ([17]) is a well-known solution concept in cooperative game theory. It shows a vector whose elements are players' share derived from several reasonable bases. There are several branches of solutions for multi-choice games that are extensions of the Shapley value. Here we apply the solution for these games proposed by Hsiao and Raghavan [5,6], which we name as the *H&R Shapley value*.

Several characterizations from TU game theory are as follows. Hart and Mas-Colell [4] introduced the potential approach to TU games. In consequence, they proved that the Shapley value can be formulated as the vector of marginal contributions of a potential. Also, the potential approach was showed to yield a characterization for the Shapley value, particularly in terms of an internal consistency property. Subsequent to this remarkable work, Ortmann [13,14] and Calvo and Santos [1] characterized the family of all solutions for TU games that admit a potential. Precisely, Ortmann [13,14] demonstrated the equivalent relations between the potentializability of a solution, the balanced contributions property and the path independence property. Calvo and Santos [1] showed that any solution that admits a potential turns out to be the Shapley value of an auxiliary game. Our results are closely related to these results.

Consistency, originally introduced by Harsanyi [3] under the name of bilateral equilibrium, is a crucial property of solutions in the axiomatic formulation of standard games. Consistency allows us to deduce, from the desirability of an outcome for some problem, the desirability of its restriction to each subgroup for the associated reduced game the subgroup faces. If a solution is not consistent, then a subgroup of agents might not respect the original compromise but revise the payoff distribution within the subgroup. The fundamental property of solutions has been investigated in various classes of problems by applying reduced games always. Various definitions of a reduced game have been proposed, depending upon exactly how the agents outside of the subgroup should be paid off. Sobolev [18] and Peleg [15,16] axiomatized the prenucleolus, the prekernel and the core, respectively, by means of consistency which respect to the reduced game due to Davis and Maschler [2]. Moulin [11] introduced an alternative version of a reduced game in the context of quasi-liner cost allocation problems. Hart and Mas-Colell [4] introduced a version of a reduced game to axiomatize the Shapley value, and so on. For discussion of this axiom, please see Thomson [19].

There are two important factors, the players and their activity levels, for multi-choice games. Inspired by Hart and Mas-Colell [4], Hsiao et al. [7] introduced the level reduced game by reducing the number of the activity levels to characterize the H&R Shapley value. Inspired by Davis and Maschler [2], Hwang and Liao [9] introduced the max-reduced game by reducing the number of the players to characterize the multi-efficient core proposed by Hwang and Liao [9].

In this paper, we continue and develop the works of Hart and Mas-Colell [4], Ortmann [13,14] and Calvo and Santos [1] on multi-choice games. The main results in this paper are as follows.

1. In Sect. 3, we show that the H&R Shapley value can be formulated as the vector of marginal contributions of a potential function.
2. In Sect. 4, we characterize the family of all solutions for multi-choice games that admit a potential, and show that any solution that admits a potential turns out to be the H&R

Shapley value of an auxiliary game. After raising the condition of independence of individual expansions, we provide the equivalent relations among the potentializability of a solution, the properties of balanced contributions and path independence.
3. In Sect. 5, by reducing the number of the players, we introduce an extension of the reduced game introduced by Hart and Mas-Colell [4] and define related property of consistency on multi-choice games. Different from the potential approach of Hart and Mas-Colell [4], we show that the H&R Shapley value satisfies related property of consistency based on "dividend".
4. Different from the axiomatizations of Hsiao and Raghavan [5,6] and Hsiao et al. [7], we provide two different types of axiomatizations in Sect. 6. One is the analogue of Myerson's [12] axiomatization of the Shapley value by applying the property of balanced contributions. The other is obtained by means of the property of consistency.

2 Definitions and notations

Let U be the universe of players and $N \subseteq U$ be a set of players and suppose each player $i \in N$ has $m_i + 1 \in \mathbb{N}$ activity levels at which he can play. Let $m = (m_i)_{i \in N}$ be the vector that describes the number of activity levels for each player, at which he can actively participate. For $i \in U$, we set $M_i = \{0, 1, \ldots, m_i\}$ as the action space of player i, where the action 0 means not participating, and $M_i^+ = M_i \backslash \{0\}$. For $N \subseteq U$, $N \neq \emptyset$, let $M^N = \prod_{i \in N} M_i$ be the product set of the action spaces for players N. Denote 0_N the zero vector in $\mathbb{R}^N$.

A multi-choice game is a triple (N, m, v), where N is a non-empty and finite set of players, m is the vector that describes the number of activity levels for each player, and $v : M^N \to \mathbb{R}$ is a characteristic function which assigns to each action vector $x = (x_i)_{i \in N} \in M^N$ the worth that the players can obtain when each player i plays at activity level $x_i \in M_i$ with $v(0_N) = 0$. If no confusion can arise a game (N, m, v) will sometimes be denoted by its characteristic function v.

Denote the class of all multi-choice games by MC. Given $(N, m, v) \in MC$ and $x \in M^N$, we write (N, x, v) for the multi-choice subgame obtained by restricting v to $\{y \in M^N \mid y_i \leq x_i \ \forall i \in N\}$ only.

Given $(N, m, v) \in MC$, let $L^{N,m} = \{(i, j) \mid i \in N, j \in M_i^+\}$. A solution on MC is a map ψ assigning to each $(N, m, v) \in MC$ an element

$$\psi(N, m, v) = \left(\psi_{i,j}(N, m, v)\right)_{(i,j) \in L^{N,m}} \in \mathbb{R}^{L^{N,m}}.$$

Here $\psi_{i,j}(N, m, v)$ is the power index or the value of the player i when he takes action j to play game v. For convenience, given $(N, m, v) \in MC$ and a solution ψ on MC, we define that for all $i \in N$, $\psi_{i,0}(N, m, v) = 0$.

To state the H&R Shapley value, some more notations will be needed. Given $S \subseteq N$, let $|S|$ be the number of elements in S and let $e^S(N)$ be the binary vector in $\mathbb{R}^N$ whose component $e_i^S(N)$ satisfies

$$e_i^S(N) = \begin{cases} 1 & \text{if } i \in S, \\ 0 & \text{otherwise.} \end{cases}$$

Note that if no confusion can arise $e_i^S(N)$ will be denoted by e_i^S.

Given $(N, m, v) \in MC$, $x \in M^N$ and $i \in N$, we define $\|x\| = \sum_{k \in N} x_k$, $S(x) = \{k \in N \mid x_k \neq 0\}$ and $A_i(x) = \{j \mid x_j \neq m_j, j \neq i\}$.

Definition 1 The *H&R Shapley value* γ is the solution on MC which associates with each $(N, m, v) \in MC$ and each $(i, j) \in L^{N,m}$ the value[1]

$$\gamma_{i,j}(v) = \sum_{k=1}^{j} \sum_{\substack{x_i=k \\ x \in M^N}} \left[\sum_{T \subseteq A_i(x)} (-1)^{|T|} \frac{1}{|S(x)| + |T \setminus S(x)|} \right] [v(x) - v(x - e^{\{i\}})].$$

Hwang and Liao [8] provided a representation of the H&R Shapley value by "dividends". The analogue of unanimity games for multi-choice games are minimal effort games (N, m, u_N^x), where $x \in M^N$, $x \neq 0$, defined by for all $y \in M^N$,

$$u_N^x(y) = \begin{cases} 1 & \text{if } y_i \geq x_i \quad \text{for all } i \in N; \\ 0 & \text{otherwise.} \end{cases}$$

Hsiao and Raghavan [5,6] showed that for $(N, m, v) \in MC$ it holds that $v = \sum_{\substack{x \in M^N \\ x \neq 0_N}} a^x \, u_N^x$, where $a^x = \sum_{S \subseteq S(x)} (-1)^{|S|} \, v(x - e^S)$.

For $(N, m, v) \in MC$, the representation of $\gamma_{i,j}(N, m, v)$ with respect to "dividend" is given by for all $(i, j) \in L^{N,m}$,

$$\gamma_{i,j}(N, m, v) = \sum_{\substack{x \in M^N \\ 0 < x_i \leq j}} \frac{a^x}{|S(x)|}.$$

The dividend a^x is the so-called divided equally among the necessary players.

3 Potential

The potential approach is a successful tool in physics. For example, a vector field G is called "conservative" if there exists a differentiable function g such that G is the gradient of g. The function g is called the potential function for G. Many important vector fields, including gravitational fields and electric force fields, are conservative. The term "conservative" is derived from the classic physical law regarding the conservation of energy. This law states that the sum of the kinetic energy and the potential energy of a particle moving in a conservative force field is constant.

In this section, we show that there exists a unique potential function on MC and moreover the vector of marginal contributions of all players (according to this potential) coincides with the H&R Shapley value.

For $x \in \mathbb{R}^N$, we write x_S to be the restriction of x at S for each $S \subseteq N$. Let $N \subseteq U$, $i \in N$ and $x \in \mathbb{R}^N$, for convenience we introduce the substitution notation x_{-i} to stand for $x_{N \setminus \{i\}}$ and let $y = (x_{-i}, j) \in \mathbb{R}^N$ be defined by $y_{-i} = x_{-i}$ and $y_i = j$. Moreover, let $p \in N$ and $l \in M_p$, x_{-ip} to stand for $x_{N \setminus \{i,p\}}$ and (x_{-ip}, j, l) to stand for $((x_{-i}, j)_{-p}, l)$.

Given a function $P : MC \longrightarrow \mathbb{R}$ which associates a real number $P(N, m, v)$ to each $(N, m, v) \in MC$. For each $(i, j) \in L^{N,m}$, we define

$$D^{i,j} P(N, m, v) = P\left(N, (m_{-i}, j), v\right) - P\left(N, (m_{-i}, 0), v\right). \tag{1}$$

[1] Hsiao and Raghavan [5,6] restricted themselves to multi-choice games where all players have the same number of activity levels and defined Shapley values using weights on the action, thereby extending ideas of weighted Shapley values (cf. [10]). We only consider the symmetric case in this paper. Indeed, the weighted case is an analog of the symmetric case.

We will use an analogue of the definition of potential stated by Ortmann ([13], Definition 3.4):

Definition 2 A solution ψ on MC admits a *potential* if there exists a function $P_\psi : MC \to \mathbb{R}$ satisfies for all $(N, m, v) \in MC$, for all $N \neq \emptyset$ and for all $(i, j) \in L^{N,m}$,

$$\psi_{i,j}(N, m, v) = D^{i,j} P(N, m, v).$$

Solutions that admit a potential assign a scalar evaluation to each game in such a way that a player's payoff is his marginal contribution to this evaluation. Moreover, a function $P : MC \longrightarrow \mathbb{R}$ is said to be *0-normalized* if for each $N \subseteq U$, $P(N, 0_N, v) = 0$. And we say it is *efficient* if it satisfies the following condition: For all $(N, m, v) \in MC$,

$$\sum_{i \in S(m)} D^{i,m_i} P(N, m, v) = v(m). \tag{2}$$

Hart and Mas-Colell [4] were the first to introduce the potential approach in cooperative transferable utility games. The following theorem is an extension of Theorem A in Hart and Mas-Colell [4]. The arguments are adaptations of the original proofs of Hart and Mas-Colell [4].

Theorem 1 *A solution ψ on MC admits a uniquely 0-normalized and efficient potential P if and only if ψ is the H&R Shapley value γ on MC. For each $(N, m, v) \in MC$ and for each $(i, j) \in L^{N,m}$*

$$\gamma_{i,j}(N, m, v) = D^{i,j} P(N, m, v).$$

Proof From Eqs. 1 and 2, it's easy to see that Eq. 2 can be rewritten as

$$P(N, m, v) = \frac{1}{|S(m)|}\left[v(m) + \sum_{i \in S(m)} P\left(N, (m_{-i}, 0), v\right)\right]. \tag{3}$$

Starting with $P(N, 0_N, v) = 0$, it determines $P(N, m, v)$ recursively. This proves the existence of the potential P, and moreover that $P(N, m, v)$ is uniquely determined by Eq. 2 (or Eq. 3) applied to (N, x, v) for all $x \in M^N$. Let

$$P(N, m, v) = \sum_{\substack{x \in M^N \\ x \neq 0_N}} \frac{a^x}{|S(x)|}. \tag{4}$$

It is easily checked that Eq. 2 is satisfied by this P; hence Eq. 4 defines the uniquely 0-normalized and efficient potential. The result now follows since for all $(i, j) \in L^{N,m}$,

$$\gamma_{i,j}(N, m, v) = \sum_{\substack{x \in M^N \\ 0 < x_i \leq j}} \frac{a^x}{|S(x)|}.$$

This completes the proof. □

4 The equivalence theorem

In this section, we characterize the family of all solutions for multi-choice TU games that admit a potential, and show that any solution that admits a potential turns out to be the multi-choice Shapley value of an auxiliary game. Also, under the condition of independence of individual expansions, we provide the equivalent relations among the potentializability of a solution, the properties of balanced contributions and path independence. To state the equivalence theorem, some more definitions will be needed.

Definition 3 Let ψ be a solution on MC.

- Efficiency (EFF)[2]: For all $(N, m, v) \in MC$, $\sum\limits_{i \in S(m)} \psi_{i,m_i}(N, m, v) = v(m)$.
- Balanced contributions (BC): For all $(N, m, v) \in MC$ and for all $(i, k_i), (j, k_j) \in L^{N,m}, i \neq j$,

$$\begin{aligned}&\psi_{i,k_i}\left(N, (m_{-j}, k_j), v\right) - \psi_{i,k_i}\left(N, (m_{-j}, 0), v\right)\\&\quad = \psi_{j,k_j}\left(N, (m_{-i}, k_i), v\right) - \psi_{j,k_j}\left(N, (m_{-i}, 0), v\right).\end{aligned}$$

- Independence of individual expansions (IIE) if for all $(N, m, v) \in MC$ and for all $(i, j) \in L^{N,m}$, $j \neq m_i$,

$$\psi_{i,j}\left(N, (m_{-i}, j), v\right) = \psi_{i,j}\left(N, (m_{-i}, j+1), v\right) = \cdots = \psi_{i,j}(N, m, v).$$

Some considerable weakenings of the previous definitions are as follows. Weak efficiency (WEFF) simply says that for all $(N, m, v) \in MC$ with $|S(m)| = 1$, ψ satisfies EFF. Upper balanced contributions (UBC) only requires that BC holds if $k_i = m_i$ and $k_j = m_j$. Weak independence of individual expansions (WIIE) simply says that for all $(N, m, v) \in MC$ with $|S(m)| = 1$, ψ satisfies IIE.

Inspired by Myerson's [12] axiomatization, we apply the balanced contributions property to a multi-choice TU game. For any two players i, j and their activity levels k_i, k_j, the payoff for the activity level k_i of player i will arise difference when player j gets available the activity level k_j and player j retires from the game. Vice versa, the payoff for the activity level k_j of player j will arise difference when player i gets available the activity level k_i and player i retires from the game. What BC asserts is that the differences of payoff will be equal. IIE asserts that whenever a player gets available higher activity level the payoff for all original levels is not changed under the condition that other players are fixed.

In the framework of TU games, Ortmann [13] offered a characterization of the potential by means of the path independence property. This characterization has its analogue in multi-choice games. To see this, we introduce the following notation.

An admissible order for $(N, m, v) \in MC$ is a bijection $\sigma : L^{N,m} \to \{1, \ldots, \sum_{i \in N} m_i\}$ satisfying[3] $\sigma(i, j) < \sigma(i, j+1)$ for all $i \in N$ and $j \in \{1, \ldots, m_i - 1\}$. The number of admissible orders for (N, m, v) is $\frac{(\sum_{i \in N} m_i)!}{\prod_{i \in N}(m_i!)}$. Let σ, σ' be admissible orders for (N, m, v), we say that σ' is a *transposition* of σ if there exist two adjacent numbers $\sigma(i, h)$ and $\sigma(j, k)$, where $(i, h), (j, k) \in L^{N,m}$ with $i \neq j$, such that σ' is obtained from σ by only switching the two adjacent numbers, i.e., there exist $(i, h), (j, k) \in L^{N,m}$ with $i \neq j$ and $\sigma(j, k) = \sigma(i, h) + 1$, such that $\sigma'(i, h) = \sigma(j, k)$, $\sigma'(j, k) = \sigma(i, h)$ and $\sigma'(p, q) = \sigma(p, q)$ for all $(p, q) \in L^{N,m} \setminus \{(i, h), (j, k)\}$. Then it is well-known that every admissible order can be transformed to another admissible order by applying transpositions.

Now let σ be an admissible order and let $k \in \{1, \ldots, \sum_{i \in N} m_i\}$. The action vector that is present after k steps according to σ, denoted by $s^{\sigma,k}$, is given by for all $i \in N$,

$$s_i^{\sigma,k} = \max\left(\{j \in M_i^+ \mid \sigma(i, j) \leq k\} \cup \{0\}\right).$$

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[2] The efficiency property was first introduced by Hsiao and Raghavan [5].

[3] For convenience, $\sigma(i, j)$ instead of $\sigma(i, j)$.

Definition 4 A solution ψ on MC satisfies path independence (PI) if for all $(N, m, v) \in MC$ and all admissible orders σ, σ' for (N, m, v)

$$\sum_{i\in N}\sum_{j=1}^{m_i}[\psi_{i,j}(N, s^{\sigma,\sigma(i,j)}, v) - \psi_{i,j-1}(N, s^{\sigma,\sigma(i,j)}, v)]$$
$$= \sum_{i\in N}\sum_{j=1}^{m_i}[\psi_{i,j}(N, s^{\sigma',\sigma'(i,j)}, v) - \psi_{i,j-1}(N, s^{\sigma',\sigma'(i,j)}, v)].$$

Definition 5 Given a solution ψ on MC and a game $(N, m, v) \in MC$, we define the auxiliary multi-choice TU game (N, m, v_ψ) as follows: For all $x \in M^N$,

$$v_\psi(x) = \sum_{i\in S(x)} \psi_{i,x_i}(N, x, v).$$

Note that if ψ satisfies efficiency then $v = v_\psi$. Now, we state the main result in this section.

Theorem 2 *Let ψ be a solution on MC. The following are equivalent :*

1. *ψ admits a potential*
2. *ψ satisfies BC and WIIE*
3. *ψ satisfies UBC and IIE*
4. *ψ satisfies PI and IIE*
5. *For all $(N, m, v) \in MC$, $\psi(N, m, v) = \gamma(N, m, v_\psi)$.*

Proof See the Appendix. □

In the framework of TU games, Theorem A in Calvo and Santos [1] presents that any solution that admits a potential turns out to be the Shapley value of an auxiliary game. And Corollary 3.4 in Calvo and Santos [1] shows that the equivalent relations among the potentializability of a solution, the properties of balanced contributions and path independence. In the framework of multi-choice games, Theorem 2 presents that any solution that admits a potential turns out to be the H&R Shapley value of an auxiliary game. Hence, one might want to characterize the family of all solutions that admit a potential using the (upper) balanced contributions property or the path independence property. However, Theorem 2 shows that besides the (upper) balanced contributions property or the path independence property, we need a third property, the (weak) independence of individual expansions property.

5 Consistency

The "consistency" requirement may be described informally as follows: Let ψ be a solution that associates a payoff to every activity level of player in every game. For any group of players in a game, one defines a "reduced game" among them by considering the amounts remaining after the rest of the players are given the payoffs prescribed by ψ. Then ψ is said to be *consistent* if, when it is applied to any reduced game, it always yields the same payoffs as in the original game.

Formally, given a solution ψ, $(N, m, v) \in MC$, and $S \subseteq N$, the reduced game $(S, m_S, v^\psi_{S,m})$ with respect to ψ, S and m is defined by

$$v^\psi_{S,m}(x) = v(x, m_{N\setminus S}) - \sum_{i\in N\setminus S} \psi_{i,m_i}\left(N, (x, m_{N\setminus S}), v\right),$$

for all $x \in M^S$. The definition and discussion follow closely the approach in Hart and Mas-colell [4].

A solution ψ on MC satisfies consistency (CON) if for all $(N, m, v) \in MC$, for all $S \subseteq N$, for all $i \in S$ and for all $j \in M_i^+$,

$$\psi_{i,j}(S, m_S, v^{\psi}_{S,m}) = \psi_{i,j}(N, m, v).$$

It is known that each $(N, m, v) \in MC$ can be expressed as a linear combination of minimal effort games and this decomposition exists uniquely. The following lemma relates the relation of coefficients of expressions between (N, m, v) and $(S, m_S, v^{\gamma}_{S,m})$.

Lemma 1 *Let $(N, m, v) \in MC, \emptyset \neq S \subseteq N$. Let $\left(S, m_S, v^{\gamma}_{S,m}\right)$ be the reduced game of (N, m, v) with respect to γ, S and m. If $v = \sum_{\substack{x \in M^N \\ x \neq 0_N}} a^x(v) \cdot u^x_N$, then $v^{\gamma}_{S,m}$ can be expressed to be*

$$v^{\gamma}_{S,m} = \sum_{\substack{y \leq m_S \\ y \neq 0_S}} a^y\left(v^{\gamma}_{S,m}\right) \cdot u^y_S,$$

where for each $y \leq m_S$, $y \neq 0_S$,

$$a^y\left(v^{\gamma}_{S,m}\right) = \sum_{t \leq m_{S^c}} \frac{|S(y)|}{|S(y)| + |S(t)|} \cdot a^{(y,t)}(v).$$

Proof Let $(N, m, v) \in MC$, $\emptyset \neq S \subseteq N$. Let $(S, m_S, v^{\gamma}_{S,m})$ be the reduced game of (N, m, v) with respect to γ, S and m. For each $y \leq m_S$ and $y \neq 0_S$,

$$\begin{aligned}
v^{\gamma}_{S,m}(y) &= v(y, m_{S^c}) - \sum_{k \in S^c} \gamma_{k,m_k}\left(N, (y, m_{S^c}), v\right) \\
&= \sum_{k \in S(y)} \gamma_{k,y_k}\left(N, (y, m_{S^c}), v\right) \\
&= \sum_{k \in S(y)} \sum_{\substack{z \leq (y, m_{S^c}) \\ 0 < z_k \leq y_k}} \frac{a^z(v)}{|S(z)|} \\
&= \sum_{k \in S(y)} \left[\sum_{\substack{z \leq (y, m_{S^c}) \\ z_k = 1}} \frac{a^z(v)}{|S(z)|} + \cdots + \sum_{\substack{z \leq (y, m_{S^c}) \\ z_k = y_k}} \frac{a^z(v)}{|S(z)|} \right] \\
&= \sum_{k \in S(y)} \left[\sum_{\substack{p \leq y \\ p_k = 1}} \sum_{t \leq m_{S^c}} \frac{a^{(p,t)}(v)}{|S(p)| + |S(t)|} + \cdots + \sum_{\substack{p \leq y \\ p_k = y_k}} \sum_{t \leq m_{S^c}} \frac{a^{(p,t)}(v)}{|S(p)| + |S(t)|} \right] \\
&= \sum_{p \leq y} \sum_{t \leq m_{S^c}} \frac{|S(p)|}{|S(p)| + |S(t)|} \cdot a^{(p,t)}(v).
\end{aligned}$$

Set $\bar{a}^y = \sum_{t \leq m_{S^c}} \frac{|S(y)|}{|S(y)| + |S(t)|} \cdot a^{(y,t)}(v)$, we have that $v^{\gamma}_{S,m} = \sum_{\substack{y \neq 0_S \\ y \leq m_S}} \bar{a}^y \cdot u^y_S$. That is, $a^y(v^{\gamma}_{S,m}) = \bar{a}^y = \sum_{t \leq m_{S^c}} \frac{|S(y)|}{|S(y)| + |S(t)|} \cdot a^{(y,t)}(v)$. □

Subsequently, different from the potential approach of Hart and Mas-Colell [4], we show that the H&R Shapley value satisfies related property of consistency based on "dividend".

Theorem 3 *The solution γ is consistent.*

Proof Let $(N, m, v) \in MC$ and $S \subseteq N$ with $S \neq \emptyset$. For each $i \in S$ and for each $j \in M_i^+$, by Lemma 1,

$$
\begin{aligned}
\gamma_{i,j}\left(S, m_S, v^{\gamma}_{S,m}\right) &= \sum_{\substack{y \le m_S \\ y_i = j}} \frac{a^y\left(v^{\gamma}_{S,m}\right)}{|S(y)|} \\
&= \sum_{\substack{y \le m_S \\ y_i = j}} \frac{1}{|S(y)|} \cdot \sum_{t \le m_{S^c}} \frac{|S(y)|}{|S(y)|+|S(t)|} \cdot a^{(y,t)}(v) \\
&= \sum_{\substack{y \le m_S \\ y_i = j}} \sum_{t \le m_{S^c}} \frac{a^{(y,t)}(v)}{|S(y)|+|S(t)|} \\
&= \sum_{\substack{x \in M^N \\ x_i = j}} \frac{a^x(v)}{|S(x)|} \\
&= \gamma_{i,j}(N, m, v).
\end{aligned}
$$

Hence, the solution γ satisfies CON. □

6 Axiomatizations

In this section, we provide two types of axiomatizations of the H&R Shapley value. One is the analogue of Myerson's [12] axiomatization of the Shapley value by applying the property of balanced contributions. The other is obtained by means of the property of consistency.

Theorem 4 1. *A solution ψ on MC satisfies EFF, IIE, and UBC if and only if $\psi = \gamma$*
2. *A solution ψ on MC satisfies EFF, WIIE, and BC if and only if $\psi = \gamma$*

Proof By Theorem 2, (1)—(2) are equivalent to each other, hence it only needs to prove (1). It is known that γ satisfies EFF and IIE. Since γ admits a potential, γ satisfies BC by Theorem 2.

To prove uniqueness, suppose that a solution ψ on MC satisfies EFF, IIE, and UBC. Since ψ satisfies IIE and UBC, $\psi(N, m, v) = \gamma(N, m, v_{\psi})$ for all $(N, m, v) \in MC$ by Theorem 2. By the definition of v_{ψ} and EFF of ψ, $v_{\psi} = v$. Hence $\psi = \gamma$, the proof is completed. □

Subsequently, we offer an axiomatization of the H&R Shapley value by means of PI.

Theorem 5 *A solution ψ on MC satisfies EFF, IIE and PI if and only if $\psi = \gamma$*

Proof It follows from Theorems 2 and 4. □

The balanced contributions property has a stronger version—BC and a weaker version—UBC. The independence of individual expansions property and the efficiency property also have the stronger versions—IIE and EFF, and the weaker versions—WIIE and WEFF, respectively. In Theorem 4, we see that the H&R Shapley value can be characterized by means of the three properties, the balanced contributions, the independence of individual expansions and the efficiency properties. More precisely, in addition to EFF, a stronger version and a weaker version between the balanced contributions and the independence of individual expansions properties is needed to characterize the H&R Shapley value. However, this conclusion does not imply that the H&R Shapley value can be characterized by combining any two of the stronger versions and one of weaker versions among the three properties mentioned above. We define a solution on MC, ψ, assigning to each $(N, m, v) \in MC$ an element $\psi(N, m, v) = (\psi_{i,j}(N, m, v))_{(i,j)\in L^{N,m}}$ to be

$$\psi_{i,j}(N, m, v) = \begin{cases} \gamma_{i,j}(N, m, v) & \text{if } |S(m)| = 1 \\ \gamma_{i,j}\left(N, (m_i, 0_{N\setminus\{i\}}), v\right) & \text{otherwise.} \end{cases}$$

We find that ψ satisfies WEFF, IIE and BC but it violates EFF. This points out that the H&R Shapley value can "not" be characterized by a weaker version—WEFF and two stronger versions—IIE and BC. However, if in addition the consistency, the H&R Shapley value can be characterized by all weaker versions of the three properties as follows.

Lemma 2 *If a solution ψ on MC satisfies WEFF and CON then it also satisfies EFF.*

Proof Let ψ be a solution on MC satisfies WEFF and CON, and $(N, m, v) \in MC$. It is trivial for $|S(m)| = 1$ by WEFF. Assume that $|S(m)| \geq 2$. Let $j \in S(m)$, consider the reduced game $\left(\{j\}, m_j, v^{\psi}_{\{j\},m}\right)$ of (N, m, v) with respect to ψ, $\{j\}$ and m. By the definition of $v^{\psi}_{\{j\},m}$,

$$v^{\psi}_{\{j\},m}(m_j) = v(m) - \sum_{i\in N\setminus\{j\}} \psi_{i,m_i}(N, m, v).$$

Since ψ satisfies CON, for $k \in M^+_j$,

$$\psi_{j,k}(N, m, v) = \psi_{j,k}\left(\{j\}, m_j, v^{\psi}_{\{j\},m}\right).$$

Then by WEFF,

$$\psi_{j,m_i}(N, m, v) = v^{\psi}_{\{j\},m}(m_j).$$

Hence $\sum_{i\in N} \psi_{i,m_i}(N, m, v) = v(m)$, i.e., ψ satisfies EFF. □

The following lemma relates the case of removing activity levels of a player before passing to the reduced game to the case of removing activity levels of a player after the passage. We show that the order does not matter.

Lemma 3 *Given a solution ψ, $(N, m, v) \in MC$, $S \subseteq N$, and $y \in M^S$, $y \neq 0_S$. Then*

$$\left(S, y, v^{\psi}_{S,m}\right) = \left(S, y, v^{\psi}_{S,(y,m_{N\setminus S})}\right).$$

Proof It is easy to derive this result by the definitions of a subgame and a reduced game, we omit it. □

Lemma 4 *If a solution ψ on MC satisfies WIIE and CON then it also satisfies IIE.*

Proof Suppose that a solution ψ on MC satisfies WIIE and CON. Let $(N, m, v) \in MC$, $i \in S(m)$ and $j \in M^+_i$, $j \neq m_i$. Let $y^k = (m_{-i}, j + k)$ for all $k = 0, 1, 2, \ldots, m_i - j$. For all k, consider the reduced game $\left(\{i\}, j + k, v^{\psi}_{\{i\},y^k}\right)$ of the subgame (N, y^k, v) of (N, m, v) with respect to ψ, $\{i\}$ and y^k, and the reduced game $\left(\{i\}, j, v^{\psi}_{\{i\},y^0}\right)$ of the subgame (N, y^0, v)

of (N, m, v) with respect to ψ,$\{i\}$ and y^0, respectively. By Lemma 3, $\left(\{i\}, j, v^{\psi}_{\{i\},y^k}\right) = \left(\{i\}, j, v^{\psi}_{\{i\},y^0}\right)$. Hence, by WIIE and CON,

$$\begin{aligned}
\psi_{i,j}\,(N, (m_{-i}, j+k), v) &= \psi_{i,j}(N, y^k, v) \;\; (\text{by } y^k = (m_{-i}, j+k)) \\
&= \psi_{i,j}\left(\{i\}, j+k, v^{\psi}_{\{i\},y^k}\right) \quad (\text{by CON}) \\
&= \psi_{i,j}\left(\{i\}, j, v^{\psi}_{\{i\},y^k}\right) \quad (\text{by WIIE}) \\
&= \psi_{i,j}\left(\{i\}, j, v^{\psi}_{\{i\},y^0}\right) \quad (\text{by Lemma 3}) \\
&= \psi_{i,j}(N, y^0, v) \quad (\text{by CON}) \\
&= \psi_{i,j}\,(N, (m_{-i}, j), v) \;\; (\text{by } y^0 = (m_{-i}, j))
\end{aligned}$$

So, ψ satisfies IIE. □

Subsequently, we provide an axiomatization by means of the consistency property.

Theorem 6 1. *A solution ψ on MC satisfies WEFF, WIIE, UBC and CON if and only if $\psi = \gamma$*
2. *A solution ψ on MC satisfies WEFF, WIIE, BC and CON if and only if $\psi = \gamma$.*

Proof This result follows from Lemmas 2, 4, and Theorems 3, 4. □

Inspired by Hart and Mas-Colell [4], we characterize the H&R Shapley value by means of the properties of consistency and standard for two-person games.

- Standard of two-person games (ST): For all $(N, m, v) \in MC$ with $|N| = 2$, $\psi(N, m, v) = \gamma(N, m, v)$.

Remark 1 If a solution ψ satisfies ST and CON, then $\psi = \gamma$ for all $(N, m, v) \in MC$ with $|S(m)| = 1$. The proof is similar to the TU-case by adding a "dummy" player to one-person problem, this is left to the reader. Hence, if ψ satisfies ST and CON, then it also satisfies WEFF and WIIE.

Theorem 7 *A solution ψ on MC satisfies ST and CON if and only if $\psi = \gamma$.*

Proof Clearly, the solution γ satisfies ST. The remaining proofs follow from Remark 1, Lemmas 2, 4 and Theorems 3, 4, 5. □

The following examples show that each of the axioms used in Theorems 4, 5 and 6 is logically independent of the remaining axioms.

Example 1 Define a solution ψ on MC by for all $(N, m, v) \in MC$ and for all $(i, j) \in L^{N,m}$,

$$\psi_{i,j}(N, m, v) = 0.$$

It's easy to verify that ψ satisfies IIE, BC and CON, but it violates WEFF and ST.

Example 2 Define a solution ψ on MC by for all $(N, m, v) \in MC$ and for all $(i, j) \in L^{N,m}$,

$$\psi_{i,j}(N, m, v) = \begin{cases} \gamma_{i,j}(N, m, v) & \text{if } \; j = m_i \\ \gamma_{i,j}(N, m, v) - \varepsilon & \text{otherwise,} \end{cases}$$

where $\varepsilon \in \mathbb{R}\backslash\{0\}$. It's easy to verify that ψ satisfies EFF, BC and CON, but it violates WIIE.

Example 3 Define a solution ψ on MC by for all $(N, m, v) \in MC$ and for all $(i, j) \in L^{N,m}$,

$$\psi_{i,j}(N, m, v) = \frac{v(m_{-i}, j)}{|S(m_{-i}, j)|}.$$

It's easy to verify that ψ satisfies EFF, IIE and CON, but it violates UBC.

Example 4 Define a solution ψ on MC by for all $(N, m, v) \in MC$ and for all $(i, j) \in L^{N,m}$,

$$\psi_{i,j}(N, m, v) = \begin{cases} \gamma_{i,j}(N, m, v) & \text{if } \; j = m_i \\ \frac{v(m_{-i}, j)}{|S(m_{-i}, j)|} & \text{otherwise.} \end{cases}$$

It's easy to verify that ψ satisfies EFF, WIIE and UBC, but it violates CON.

Example 5 Define a solution ψ on MC by for all $(N, m, v) \in MC$ and for all $(i, j) \in L^{N,m}$,

$$\psi_{i,j}(N, m, v) = \begin{cases} \gamma_{i,j}(N, m, v) & \text{if } \; |S(m)| \leq 2 \\ \gamma_{i,j}(N, m, v) - \varepsilon & \text{otherwise,} \end{cases}$$

where $\varepsilon \in \mathbb{R}\backslash\{0\}$. It's easy to verify that σ satisfies ST, but it violates CON.

Appendix: The proof of Theorem 2

Proof Let ψ be a solution on MC. To verify 1 $\Rightarrow$ 2, suppose ψ admits a potential P. Let $(N, m, v) \in MC$ and $(i, k_i), (j, k_j) \in L^{N,m}$, $i \neq j$,

$$\begin{aligned} &\psi_{i,k_i}\left(N, (m_{-j}, k_j), v\right) - \psi_{i,k_i}\left(N, (m_{-j}, 0), v\right) \\ &\quad = \left[P\left(N, (m_{-ij}, k_i, k_j), v\right) - P\left(N, (m_{-ij}, 0, k_j), v\right)\right] \\ &\qquad - \left[P\left(N, (m_{-ij}, k_i, 0), v\right) - P\left(N, (m_{-ij}, 0, 0), v\right)\right] \\ &\quad = \left[P\left(N, (m_{-ij}, k_i, k_j), v\right) - P\left(N, (m_{-ij}, k_i, 0), v\right)\right] \\ &\qquad - \left[P\left(N, (m_{-ij}, 0, k_j), v\right) - P\left(N, (m_{-ij}, 0, 0), v\right)\right] \\ &\quad = \psi_{j,k_j}\left(N, (m_{-i}, k_i), v\right) - \psi_{j,k_j}\left(N, (m_{-i}, 0), v\right). \end{aligned}$$

Hence, ψ satisfies BC. To see that ψ satisfies WIIE, we show that it satisfies IIE. Let $(N, m, v) \in MC$ and $(i, j) \in L^{N,m}$, $j \neq m_i$. For $j \leq l \leq m_i$

$$\begin{aligned} &\psi_{i,j}\left(N, (m_{-i}, l), v\right) \\ &\quad = P\left(N, (m_{-i}, j), v\right) - P\left(N, (m_{-i}, 0), v\right) \\ &\quad = \psi_{i,j}(N, m, v). \end{aligned}$$

That is, ψ satisfies IIE.

To verify 2 $\Rightarrow$ 3, suppose ψ satisfies BC and WIIE. Clearly, ψ satisfies UBC. It remains to show that ψ satisfies IIE. Let $(N, m, v) \in MC$. The proof proceeds by induction on the number $\|m\|$. It is true for $\|m\| = 1$ by WIIE. Assume that ψ satisfies IIE for $\|m\| \leq t - 1$, where $t \geq 2$.

The case $\|m\| = t$: If $|S(m)| = 1$ then we have done by WIIE. Hence, without loss of generality, we assume that $|S(m)| \geq 2$. Two cases may be distinguished:

Case 1. If $m_i = 0$ *or* 1 for all $i \in N$:

In this situation, there is no $(i, j) \in L^{N,m}$ with $j \neq m_i$, hence we have done.

Case 2. There exist $a, b \in N$ such that $m_a \geq 2$ and $m_b \neq 0$:

For $(i, j) \in L^{N,m}$ with $j \neq m_i$, let $p \in S(m)$ and $p \neq i$. For all $k = 0, 1, 2, \ldots, m_i - j$, consider the game $\big(N, (m_{-ip}, j + k, m_p), v\big)$, by BC of ψ,

$$\begin{aligned}
&\psi_{i,j}\left(N, (m_{-ip}, j + k, m_p), v\right) - \psi_{i,j}\left(N, (m_{-ip}, j + k, 0), v\right) \\
&\quad = \psi_{p,m_p}\left(N, (m_{-ip}, j, m_p), v\right) - \psi_{p,m_p}\left(N, (m_{-ip}, 0, m_p), v\right).
\end{aligned}$$

Hence, for all $k = 0, 1, 2, \ldots, m_i - j$,

$$\begin{aligned}
&\psi_{i,j}\left(N, (m_{-ip}, j + k, m_p), v\right) \\
&\quad = \psi_{i,j}\left(N, (m_{-ip}, j + k, 0), v\right) + \psi_{p,m_p}\left(N, (m_{-ip}, j, m_p), v\right) \\
&\qquad - \psi_{p,m_p}\left(N, (m_{-ip}, 0, m_p), v\right).
\end{aligned}$$

Since $\|(m_{-ip}, j+k, 0)\| < \|m\|$, by the induction hypotheses, for all $k = 0, 1, 2, \ldots, m_i - j$,

$$\psi_{i,j}\left(N, (m_{-ip}, j, 0), v\right) = \psi_{i,j}\left(N, (m_{-ip}, j + k, 0), v\right).$$

So, for all $k = 0, 1, 2, \ldots, m_i - j$,

$$\begin{aligned}
&\psi_{i,j}\left(N, (m_{-ip}, j, m_p), v\right) \\
&\quad = \psi_{i,j}\left(N, (m_{-ip}, j, 0), v\right) + \psi_{p,m_p}\left(N, (m_{-ip}, j, m_p), v\right) \\
&\qquad - \psi_{p,m_p}\left(N, (m_{-ip}, 0, m_p), v\right) \\
&\quad = \psi_{i,j}\left(N, (m_{-ip}, j + k, 0), v\right) + \psi_{p,m_p}\left(N, (m_{-ip}, j, m_p), v\right) \\
&\qquad - \psi_{p,m_p}\left(N, (m_{-ip}, 0, m_p), v\right) \\
&\quad = \psi_{i,j}\left(N, (m_{-ip}, j + k, m_p), v\right).
\end{aligned}$$

That is, ψ satisfies IIE.

To verify $3 \Rightarrow 2$, suppose ψ satisfies UBC and IIE. It remains to show that ψ satisfies BC. Let $(N, m, v) \in MC$ and $(i, k_i), (j, k_j) \in L^{N,m}$, $i \neq j$. By IIE,

$$\begin{aligned}
&\text{(i) } \psi_{i,k_i}\left(N, (m_{-j}, k_j), v\right) = \psi_{i,k_i}\left(N, (m_{-ij}, k_i, k_j), v\right) \\
&\text{(ii) } \psi_{i,k_i}\left(N, (m_{-j}, 0), v\right) = \psi_{i,k_i}\left(N, (m_{-ij}, k_i, 0), v\right) \\
&\text{(iii) } \psi_{j,k_j}\left(N, (m_{-i}, k_i), v\right) = \psi_{j,k_j}\left(N, (m_{-ij}, k_i, k_j), v\right) \\
&\text{(iv) } \psi_{j,k_j}\left(N, (m_{-i}, 0), v\right) = \psi_{j,k_j}\left(N, (m_{-ij}, 0, k_j), v\right).
\end{aligned}$$

Using UBC to the game $\big(N, (m_{-ij}, k_i, k_j), v\big)$ and by (i)—(iv),

$$\begin{aligned}
&\psi_{i,k_i}\left(N, (m_{-j}, k_j), v\right) - \psi_{i,k_i}\left(N, (m_{-j}, 0), v\right) \\
&\quad = \psi_{i,k_i}\left(N, (m_{-ij}, k_i, k_j), v\right) - \psi_{i,k_i}\left(N, (m_{-ij}, k_i, 0), v\right) \quad \text{(by (i), (ii))} \\
&\quad = \psi_{j,k_j}\left(N, (m_{-ij}, k_i, k_j), v\right) - \psi_{j,k_j}\left(N, (m_{-ij}, 0, k_j), v\right) \quad \text{(by UBC)} \\
&\quad = \psi_{j,k_j}\left(N, (m_{-i}, k_i), v\right) - \psi_{j,k_j}\left(N, (m_{-i}, 0), v\right). \quad \text{(by (iii), (iv))}
\end{aligned}$$

Hence ψ satisfies BC.

To verify $2 \Rightarrow 4$, suppose ψ satisfies BC and WIIE, hence ψ satisfies IIE by $2 \Rightarrow 3$. It remains to show that ψ satisfies PI. Let $(N, m, v) \in MC$ and σ, σ' be two admissible orders for (N, m, v). Since every admissible order can be transformed to another admissible order by applying transpositions, we can assume that σ' is a transposition of σ. Let $(i, h), (j, k) \in L^{N,m}$ with $i \neq j$ and $\sigma(j, k) = \sigma(i, h) + 1$, such that $\sigma'(i, h) = \sigma(j, k)$, $\sigma'(j, k) = \sigma(i, h)$ and $\sigma'(p, q) = \sigma(p, q)$ for all $(p, q) \in L^{N,m} \backslash \{(i, h), (j, k)\}$.
Since σ' is a transposition of σ, we have that

(a) $\psi_{p,q}(N, s^{\sigma,\sigma(p,q)}, v) = \psi_{p,q}(N, s^{\sigma',\sigma'(p,q)}, v)$ and for all $(p, q) \in L^{N,m} \backslash \{(i, h), (j, k)\}$, $\psi_{p,q-1}(N, s^{\sigma,\sigma(p,q)}, v) = \psi_{p,q-1}(N, s^{\sigma',\sigma'(p,q)}, v)$.
(b) $\psi_{i,q}(N, s^{\sigma,\sigma(i,q)}, v) = \psi_{i,q}(N, s^{\sigma',\sigma'(i,q)}, v)$ and for all $q \in M_i^+, q \neq h$, $\psi_{i,q-1}(N, s^{\sigma,\sigma(i,q)}, v) = \psi_{i,q-1}(N, s^{\sigma',\sigma'(i,q)}, v)$.
(c) $\psi_{j,q}(N, s^{\sigma,\sigma(j,q)}, v) = \psi_{j,q}(N, s^{\sigma',\sigma'(j,q)}, v)$ and for all $q \in M_j^+, q \neq k$, $\psi_{j,q-1}(N, s^{\sigma,\sigma(j,q)}, v) = \psi_{j,q-1}(N, s^{\sigma',\sigma'(j,q)}, v)$.

Put $x = s^{\sigma,\sigma(i,h)-1} = s^{\sigma',\sigma'(j,k)-1}$, then $x_i = h - 1$ and $x_j = k - 1$. By (a)—(c),

$$
\begin{aligned}
&\sum_{p\in N}\sum_{q=1}^{m_p}\left[\psi_{p,q}(N, s^{\sigma,\sigma(p,q)}, v) - \psi_{p,q-1}(N, s^{\sigma,\sigma(p,q)}, v)\right] \\
&\quad - \sum_{p\in N}\sum_{q=1}^{m_p}\left[\psi_{p,q}(N, s^{\sigma',\sigma'(p,q)}, v) - \psi_{p,q-1}(N, s^{\sigma',\sigma'(p,q)}, v)\right] \\
&= \big\{\big[\psi_{i,h}\left(N, (x_{-i}, h), v\right) - \psi_{i,h-1}\left(N, (x_{-i}, h), v\right)\big] \\
&\quad + \big[\psi_{j,k}\left(N, (x_{-ij}, h, k), v\right) - \psi_{j,k-1}\left(N, (x_{-ij}, h, k), v\right)\big]\big\} \\
&\quad - \big\{\big[\psi_{j,k}\left(N, (x_{-j}, k), v\right) - \psi_{j,k-1}\left(N, (x_{-j}, k), v\right)\big] \\
&\quad + \big[\psi_{i,h}\left(N, (x_{-ij}, h, k), v\right) - \psi_{i,h-1}\left(N, (x_{-ij}, h, k), v\right)\big]\big\} \\
&= \big\{[\psi_{i,h}\left(N, (x_{-i}, h), v\right) - \psi_{i,h-1}\left(N, x, v\right)] \quad \text{(by IIE)} \\
&\quad + \big[\psi_{j,k}\left(N, (x_{-ij}, h, k), v\right) - \psi_{j,k-1}\left(N, (x_{-i}, h), v\right)\big]\big\} \quad \text{(by IIE)} \\
&\quad - \big\{\big[\psi_{j,k}\left(N, (x_{-j}, k), v\right) - \psi_{j,k-1}\left(N, x, v\right)\big] \quad \text{(by IIE)} \\
&\quad + \big[\psi_{i,h}\left(N, (x_{-ij}, h, k), v\right) - \psi_{i,h-1}\left(N, (x_{-j}, k), v\right)\big]\big\} \quad \text{(by IIE)} \\
&= \big[\psi_{i,h}\left(N, (x_{-i}, h), v\right) - \psi_{j,k-1}\left(N, (x_{-i}, h), v\right)\big] \\
&\quad - \big[\psi_{i,h-1}\left(N, x, v\right) - \psi_{j,k-1}\left(N, x, v\right)\big] \\
&\quad + \big[\psi_{j,k}\left(N, (x_{-ij}, h, k), v\right) - \psi_{ih}\left(N, (x_{-ij}, h, k), v\right)\big] \\
&\quad - \big[\psi_{j,k}\left(N, (x_{-j}, k), v\right) - \psi_{i,h-1}\left(N, (x_{-j}, k), v\right)\big].
\end{aligned} \tag{5}
$$

If $h = k = 1$ then $x_i = x_j = 0$, we have that

$$
\begin{aligned}
(5) &= \psi_{i,1}\left(N, (x_{-ij}, 1, 0), v\right) + \psi_{j,1}\left(N, (x_{-ij}, 1, 1), v\right) \\
&\quad - \psi_{i,1}\left(N, (x_{-ij}, 1, 1), v\right) - \psi_{j,1}\left(N, (x_{-ij}, 0, 1), v\right) \\
&= \psi_{i,1}\left(N, (x_{-ij}, 1, 0), v\right) - \psi_{i,1}\left(N, (x_{-ij}, 1, 1), v\right) \\
&\quad + \psi_{j,1}\left(N, (x_{-ij}, 1, 1), v\right) - \psi_{j,1}\left(N, (x_{-ij}, 0, 1), v\right) \qquad \text{(by BC)} \\
&= 0.
\end{aligned}
$$

If $h = 1$ and $k > 1$ then $x_i = 0$, we have that

$$
\begin{aligned}
(5) &= \psi_{i,1}\left(N, (x_{-i}, 1), v\right) - \psi_{j,k-1}\left(N, (x_{-i}, 1), v\right) \\
&\quad + \psi_{j,k-1}\left(N, x, v\right) + \psi_{j,k}\left(N, (x_{-ij}, 1, k), v\right) \\
&\quad - \psi_{i,1}\left(N, (x_{-ij}, 1, k), v\right) - \psi_{j,k}\left(N, (x_{-j}, k), v\right) \\
&= \{\psi_{i,1}\left(N, (x_{-i}, 1), v\right) - \psi_{j,k-1}\left(N, (x_{-i}, 1), v\right) + \psi_{j,k-1}\left(N, x, v\right)\} \\
&\quad + \left\{\psi_{j,k}\left(N, (x_{-ij}, 1, k), v\right) - \psi_{i,1}\left(N, (x_{-ij}, 1, k), v\right) - \psi_{j,k}\left(N, (x_{-j}, k), v\right)\right\} \\
&= \psi_{i,1}\left(N, (x_{-ij}, 1, 0), v\right) - \psi_{i,1}\left(N, (x_{-ij}, 1, 0), v\right) \qquad \text{(by BC)} \\
&= 0.
\end{aligned}
$$

If $h > 1$ and $k = 1$ then the proof is similar to the case of $h = 1, k > 1$.

If $h > 1$ and $k > 1$ then we have that

$$\begin{aligned}
(5) = &\left[\psi_{i,h}\left(N,(x_{-i},h),v\right) - \psi_{j,k-1}\left(N,(x_{-i},h),v\right)\right] \\
&- \left[\psi_{i,h-1}\left(N,x,v\right) - \psi_{j,k-1}\left(N,x,v\right)\right] \\
&+ \left[\psi_{j,k}\left(N,(x_{-ij},h,k),v\right) - \psi_{i,h}\left(N,(x_{-ij},h,k),v\right)\right] \\
&- \left[\psi_{j,k}\left(N,(x_{-j},k),v\right) - \psi_{i,h-1}\left(N,(x_{-j},k),v\right)\right] \\
= &\left[\psi_{i,h}\left(N,(x_{-ij},h,0),v\right) - \psi_{j,k-1}\left(N,(x_{-i},0),v\right)\right] \quad \text{(by BC)} \\
&- \left[\psi_{i,h-1}\left(N,(x_{-j},0),v\right) - \psi_{j,k-1}\left(N,(x_{-i},0),v\right)\right] \quad \text{(by BC)} \\
&+ \left[\psi_{j,k}\left(N,(x_{-ij},0,k),v\right) - \psi_{i,h}\left(N,(x_{-ij},h,0),v\right)\right] \quad \text{(by BC)} \\
&- \left[\psi_{j,k}\left(N,(x_{-ij},0,k),v\right) - \psi_{i,h-1}\left(N,(x_{-j},0),v\right)\right] \quad \text{(by BC)} \\
= &\ 0.
\end{aligned}$$

Hence, ψ satisfies PI.

To verify $4 \Rightarrow 5$, suppose ψ satisfies PI and IIE. Let $(N, m, v) \in MC$. The proof proceeds by induction on the number $\|m\|$. If $\|m\| = 1$, let $S(m) = \{i\}$ and $m_i = 1$, then by the definition of v_ψ and efficiency of γ,

$$\psi_{i,1}(N,m,v) = v_\psi(m) = \gamma_{i,1}(N,m,v_\psi).$$

Suppose that $\psi(N, m, v) = \gamma(N, m, v_\psi)$ for $\|m\| \le k$, where $k \ge 1$.

The case $\|m\| = k+1$: For every $(h,l) \in L^{N,m}$, $l \ne m_h$, by IIE and induction hypotheses,

$$\begin{aligned}
\psi_{h,l}(N,m,v) &= \psi_{h,l}\left(N,(m_{-h},l),v\right) \\
&= \gamma_{h,l}\left(N,(m_{-h},l),v_\psi\right) \\
&= \gamma_{h,l}\left(N,m,v_\psi\right).
\end{aligned} \tag{6}$$

Hence it remains to show that $\psi_{h,m_h}(N,m,v) = \gamma_{h,m_h}(N,m,v_\psi)$ for all $h \in N$. For every $(h, m_h) \in L^{N,m}$, let σ_h be an admissible order with $\sigma_h(h,m_h) = \|m\| = k+1$. Since ψ satisfies PI, for all $h, h' \in N$,

$$\begin{aligned}
&\psi_{h,m_h}(N,m,v) - \psi_{h,m_{h-1}}(N,m,v) \\
&\quad + \sum_{j=1}^{m_h-1}\left[\psi_{h,j}(N,s^{\sigma_h,\sigma_h(h,j)},v) - \psi_{h,j-1}(N,s^{\sigma_h,\sigma_h(h,j)},v)\right] \\
&\quad + \sum_{i\in N\setminus\{h\}}\sum_{j=1}^{m_i}\left[\psi_{i,j}(N,s^{\sigma_h,\sigma_h(i,j)},v) - \psi_{i,j-1}(N,s^{\sigma_h,\sigma_h(i,j)},v)\right] \\
&= \psi_{h,m_{h'}}(N,m,v) - \psi_{h,m_{h'-1}}(N,m,v) \\
&\quad + \sum_{j=1}^{m_{h'}-1}\left[\psi_{h',j}(N,s^{\sigma_{h'},\sigma_{h'}(h',j)},v) - \psi_{h',j-1}(N,s^{\sigma_{h'},\sigma_{h'}(h',j)},v)\right] \\
&\quad + \sum_{i\in N\setminus\{h'\}}\sum_{j=1}^{m_i}\left[\psi_{i,j}(N,s^{\sigma_{h'},\sigma_{h'}(i,j)},v) - \psi_{i,j-1}(N,s^{\sigma_{h'},\sigma_{h'}(i,j)},v)\right].
\end{aligned} \tag{7}$$

Since the H&R Shapley value γ admits a potential, γ satisfies PI. So, for all $h, h' \in N$,

$$\gamma_{h,m_h}(N,m,v_\psi) - \gamma_{h,m_{h-1}}(N,m,v_\psi)$$

$$
\begin{aligned}
&+\sum_{j=1}^{m_h-1}\Big[\gamma_{h,j}(N,s^{\sigma_h,\sigma_h(h,j)},v_\psi)-\gamma_{h,j-1}(N,s^{\sigma_h,\sigma_h(h,j)},v_\psi)\Big]\\
&+\sum_{i\in N\setminus\{h\}}\sum_{j=1}^{m_i}\Big[\gamma_{i,j}(N,s^{\sigma_h,\sigma_h(i,j)},v_\psi)-\gamma_{i,j-1}(N,s^{\sigma_h,\sigma_h(i,j)},v_\psi)\Big]\\
=\,&\gamma_{h,m_{h'}}(N,m,v_\psi)-\gamma_{h,m_{h'-1}}(N,m,v_\psi)\\
&+\sum_{j=1}^{m_{h'}-1}\Big[\gamma_{h',j}(N,s^{\sigma_{h'},\sigma_{h'}(h',j)},v_\psi)-\gamma_{h',j-1}(N,s^{\sigma_{h'},\sigma_{h'}(h',j)},v_\psi)\Big]\\
&+\sum_{i\in N\setminus\{h'\}}\sum_{j=1}^{m_i}\Big[\gamma_{i,j}(N,s^{\sigma_{h'},\sigma_{h'}(i,j)},v_\psi)-\gamma_{i,j-1}(N,s^{\sigma_{h'},\sigma_{h'}(i,j)},v_\psi)\Big]. \qquad (8)
\end{aligned}
$$

For every $h \in N$, let

$$
\begin{aligned}
\mathcal{C}_h^\psi =& -\psi_{h,m_{h-1}}(N,m,v)+\sum_{j=1}^{m_h-1}\Big[\psi_{h,j}(N,s^{\sigma_h,\sigma_h(h,j)},v)-\psi_{h,j-1}(N,s^{\sigma_h,\sigma_h(h,j)},v)\Big]\\
&+\sum_{i\in N\setminus\{h\}}\sum_{j=1}^{m_i}\Big[\psi_{i,j}(N,s^{\sigma_h,\sigma_h(i,j)},v)-\psi_{i,j-1}(N,s^{\sigma_h,\sigma_h(i,j)},v)\Big],
\end{aligned}
$$

and

$$
\begin{aligned}
\mathcal{C}_h^\gamma =& -\gamma_{h,m_{h-1}}(N,m,v_\psi)+\sum_{j=1}^{m_h-1}\Big[\gamma_{h,j}(N,s^{\sigma_h,\sigma_h(h,j)},v_\psi)-\gamma_{h,j-1}(N,s^{\sigma_h,\sigma_h(h,j)},v_\psi)\Big]\\
&+\sum_{i\in N\setminus\{h\}}\sum_{j=1}^{m_i}\Big[\gamma_{i,j}(N,s^{\sigma_h,\sigma_h(i,j)},v_\psi)-\gamma_{i,j-1}(N,s^{\sigma_h,\sigma_h(i,j)},v_\psi)\Big].
\end{aligned}
$$

Since $\|s^{\sigma_h,\sigma_h(i,j)}\| \le k$ for $(i, j) \ne (h, m_h)$, by (6) and the induction hypotheses, we see that for all $h \in N$, $\mathcal{C}_h^\psi = \mathcal{C}_h^\gamma$. Let $\mathcal{C}_h = \mathcal{C}_h^\psi = \mathcal{C}_h^\gamma$ for all $h \in N$, hence, Eqs. 7 and 8 become

$$\psi_{h,m_h}(N,m,v)+\mathcal{C}_h=\psi_{h,m_{h'}}(N,m,v)+\mathcal{C}_{h'}, \qquad (9)$$

and

$$\gamma_{h,m_h}(N,m,v_\psi)+\mathcal{C}_h=\gamma_{h,m_{h'}}(N,m,v_\psi)+\mathcal{C}_{h'}. \qquad (10)$$

Combining Eqs. 9 with 10, we obtain that for all $h, h' \in N$,

$$\psi_{h,m_h}(N,m,v)-\gamma_{h,m_h}(N,m,v_\psi)=\psi_{h,m_{h'}}(N,m,v)-\gamma_{h,m_{h'}}(N,m,v_\psi),$$

Let $d = \psi_{h,m_h}(N, m, v) - \gamma_{h,m_h}(N, m, v_\psi)$ for all $h \in N$. By the definition of v_ψ and the efficiency of γ,

$$
\begin{aligned}
|N|\cdot d &= \textstyle\sum_{h\in N}\psi_{h,m_h}(N,m,v)-\sum_{h\in N}\gamma_{h,m_h}(N,m,v_\psi)\\
&=v_\psi(m)-v_\psi(m)\\
&=0.
\end{aligned}
$$

Hence, for all $h \in N$, $d = \psi_{h,m_h}(N, m, v) - \gamma_{h,m_h}(N, m, v_\psi) = 0$. That is, for all $h \in N$, $\psi_{h,m_h}(N, m, v) = \gamma_{h,m_h}(N, m, v_\psi)$. Hence $\psi(N, m, v) = \gamma(N, m, v_\psi)$.

To verify 5 $\Rightarrow$ 1, suppose that $\psi(N, m, v) = \gamma(N, m, v_\psi)$ for all $(N, m, v) \in MC$. Since the H&R Shapley value γ on MC admits a potential P_γ, we define a function of ψ as $P_\psi(N, m, v) = P_\gamma(N, m, v_\psi)$ for all $(N, m, v) \in MC$. Then for every $(i, j) \in L^{N,m}$,

$$\begin{aligned}
&P_\psi\left(N, (m_{-i}, j), v\right) - P_\psi\left(N, (m_{-i}, 0), v\right)\\
&\quad = P_\gamma\left(N, (m_{-i}, j), v_\psi\right) - P_\gamma\left(N, (m_{-i}, 0), v_\psi\right)\\
&\quad = \gamma_{i,j}(N, m, v_\psi)\\
&\quad = \psi_{i,j}(N, m, v).
\end{aligned}$$

By Definition 2, the solution ψ admits a potential P_ψ. □

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References

1. Calvo, E., Santos, J.C.: Potential in cooperative TU-games. Math. Soc. Sci. **34**, 175–190 (1997)
2. Davis, M., Maschler, M.: The kernel of a cooperative game. Nav. Res. Logist. Q. **12**, 223–259 (1965)
3. Harsanyi, J.C.: A bargaining model for the cooperative n-person game. In: Tucker, A.W., R.D. Luce (eds.) Contributions to the Theory of Games IV. Annals of Mathematics Studies, vol. 40, pp. 325–355. Princeton University Press, Princeton (1959)
4. Hart, S., Mas-Colell, A.: Potential, value and consistency. Econometrica **57**, 589–614 (1989)
5. Hsiao, C.R., Raghavan, T.E.S.: Monotonicity and dummy free property for multi-choice cooperative games. Int. J. Game Theory **21**, 301–302 (1992)
6. Hsiao, C.R., Raghavan, T.E.S.: Shapley value for multi-choice cooperative game (I). Games Econ. Behav. **5**, 240–256 (1993)
7. Hsiao, C.R., Yeh, Y.N., Mo, J.P.: The potential of multi-choice cooperative game. Conference Paper. In: International Mathematics Conference ’94, National Sun Yat-sen University, Kaohsiung, Taiwan, 2–5 Dec, 1994
8. Hwang, Y.A., Liao, Y.H.: A comparison between the Shapley value based solutions for multi-choice games. Submitted (2007)
9. Hwang, Y.A., Liao, Y.H.: The multi-efficient core, balancedness and axiomatizations for multi-choice games. Int. J. Game Theory. Revised (2008)
10. Kalai, E., Samet, D. : Weighted Shapley values. In: Roth, A.E. (ed.) The Shapley Value, pp. 83–99. Cambridge University Press, New York (1988)
11. Moulin, H.: The separability axiom and equal-sharing methods. J. Econ. Theory **36**, 120–148 (1985)
12. Myerson, R.: Conference structures and fair allocation rules. Int. J. Game Theory **9**, 169–182 (1980)
13. Ortmann, K.M.: Preservation of differences, potential, conservity. Working Paper 236, Institute of Mathematical Economics, University of Bielefeld (1995a)
14. Ortmann, K.M.: Conservation of energy in nonatomic games. Working Paper 237, Institute of Mathematical Economics, University of Bielefeld (1995b)
15. Peleg, B.: An axiomatization of the core of cooperative games without side payments. J. Math. Econ. **14**, 203–214 (1985)
16. Peleg, B.: On the reduced game property and its converse. Int. J. Game Theory **15**, 187–200 (1986)
17. Shapley, L.S.: A value for n-person game. In: Kuhn, H.W., Tucker, A.W. (eds.) Contributions to the Theory of Games II. Annals of Mathematics Studies, vol. 28, pp. 307–317. Princeton University Press, Princeton (1953)
18. Sobolev, A.I.: The characterization of optimality principles in cooperative games by functional equations. Math. Methods Soc. Sci. **6**, 150–165 (1975)
19. Thomson, W.: Consistent allocation rules. Mimeo, University of Rochester (2005)